

# A note on the Moser-Trudinger inequality in Sobolev-Slobodeckij spaces in dimension one

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## Abstract

We discuss some recent results by Parini and Ruf on a Moser-Trudinger type inequality in the setting of Sobolev-Slobodeckij spaces in dimension one. We push further their analysis considering the inequality on the whole  $\mathbb{R}$  and we give an answer to one of their open questions.

## 1 Introduction

A classical result in analysis states that, if  $\Omega \subset \mathbb{R}^n$  is an open set with finite measure  $|\Omega|$  and Lipschitz boundary,  $k$  is a positive integer with  $k < n$ , and  $p \in [1, \frac{k}{n})$ , then the Sobolev space  $W_0^{k,p}(\Omega)$  embeds continuously in  $L^{\frac{np}{n-kp}}(\Omega)$ . This results doesn't hold for the critical case  $p = \frac{n}{k}$ , that is  $W_0^{k,\frac{n}{k}}(\Omega)$  doesn't embed in  $L^\infty(\Omega)$ . On the other hand Trudinger [14], Pohozaev [12], Yudovich [6] and others found that, at least in the case  $k = 1$ , functions in  $W_0^{1,n}(\Omega)$  enjoy summability of exponential type. Namely

$$W_0^{1,n}(\Omega) \subset \left\{ u \in L^1(\Omega) : \int_{\Omega} e^{\beta|u|^{\frac{n}{n-1}}} dx < +\infty \right\}$$

for any  $\beta < +\infty$ . Moser [9] sharpened this embedding and determined the optimal exponent  $\alpha_n$  such that

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha_n |u|^{\frac{n}{n-1}}} dx < C|\Omega|, \quad \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}. \quad (1)$$

Here,  $\omega_{n-1}$  is the volume of the unit sphere in  $\mathbb{R}^n$ . In particular the exponent  $\alpha_n$  is sharp in the sense that

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha |u|^{\frac{n}{n-1}}} dx = +\infty$$

for any  $\alpha > \alpha_n$ . Moreover, the supremum in (1) becomes infinite as soon as we slightly modify the integrand, namely

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} f(|u|) e^{\alpha_n |u|^{\frac{n}{n-1}}} dx = +\infty \quad (2)$$

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for any measurable function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\lim_{t \rightarrow +\infty} f(t) = \infty$ . This can be proved, for instance, using the same test functions defined in [9]. In [1] Adams, exploiting Riesz potentials, extended Moser's result to higher order Sobolev spaces  $W_0^{k,p}(\Omega)$ ,  $k > 1$ ,  $p = \frac{n}{k}$ .

In the present work, we are interested in generalizations of (1) that concern Sobolev spaces of fractional orders. The usual approach is to consider Bessel potential spaces  $H^{s,p}$ . In this setting, sharp versions of (1) are proven both in the cases of bounded and unbounded domains of  $\mathbb{R}^n$ ,  $n \geq 1$  (see [5], [8] and [4]).

Here, we focus our attention on the case (in general different from the one of Bessel potential spaces) of Sobolev-Slobodeckij spaces (see definitions below), which has been recently proposed, together with some open questions, by Parini and Ruf. In [10] they considered  $\Omega \subset \mathbb{R}^n$  to be a bounded and open domain,  $n \geq 2$  and  $sp = n$  and they were able to prove the existence of  $\alpha_* > 0$  such that the corresponding version of inequality (1) is satisfied for any  $\alpha \in (0, \alpha_*)$  (see also [11]). Even though the result is not sharp, in the sense that the value of the optimal exponent is not yet known, an explicit upper bound for the optimal exponent  $\alpha^*$  is given.

As a first step, we extend the results in [10] to the case  $n = 1$ . For any  $s \in (0, 1)$  and  $p > 1$ , the Sobolev-Slobodeckij space  $W^{s,p}(\mathbb{R})$  is defined as

$$W^{s,p}(\mathbb{R}) := \{u \in L^p(\mathbb{R}) : [u]_{W^{s,p}(\mathbb{R})} < +\infty\}$$

where  $[u]_{W^{s,p}(\mathbb{R})}$  is the Gagliardo seminorm defined by

$$[u]_{W^{s,p}(\mathbb{R})} := \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy \right)^{\frac{1}{p}}. \quad (3)$$

We will often write  $[\cdot] := [\cdot]_{W^{s,p}(\mathbb{R})}$ . The space  $W^{s,p}(\mathbb{R})$  is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\mathbb{R})} := \left( \|u\|_{L^p(\mathbb{R})}^p + [u]_{W^{s,p}(\mathbb{R})}^p \right)^{\frac{1}{p}}. \quad (4)$$

Let  $I$  be an open interval in  $\mathbb{R}$ . We define the space  $\tilde{W}_0^{s,p}(I)$  as the closure of  $(C_0^\infty(I), \|u\|_{W^{s,p}(\mathbb{R})})$ . An equivalent definition for  $\tilde{W}_0^{s,p}(I)$  can be obtained taking the completion of  $C_0^\infty(I)$  with respect to the seminorm  $[u]_{W^{s,p}(\mathbb{R})}$  (see [3, Remark 2.5]).

With a mild adaptation of the techniques used in [10], we are able to prove that their result holds also in dimension one.

**Theorem 1.1.** *Let  $s \in (0, 1)$  and  $p > 1$  be such that  $sp = 1$ . There exists  $\alpha_* = \alpha_*(s) > 0$  such that for all  $\alpha \in [0, \alpha_*)$  it holds*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \leq 1} \int_I e^{\alpha|u|^{\frac{1}{1-s}}} dx < \infty. \quad (5)$$

Moreover, there exists  $\alpha^* = \alpha^*(s) := \gamma_s^{\frac{s}{1-s}}$  such that the supremum in (5) is infinite for any  $\alpha \in (\alpha^*, +\infty)$ .

It is worth to remark that, as already pointed out in [10], the exponent  $\alpha^*(\frac{1}{2})$  is equal to  $2\pi^2$  and it coincides, up to a normalization constant, with the optimal exponent  $\pi$  determined in [5] in the setting of Bessel potential spaces.

We move now to the case  $I = \mathbb{R}$ , pushing further the analysis of [10]. An inequality of the form (5) cannot hold if we don't consider the full  $W^{s,p}(\mathbb{R})$ -norm, i.e. we take into account also the term  $\|u\|_{L^p(\mathbb{R})}$ . This has been done by Ruf [13] in the case of  $H^{1,2}(\mathbb{R}^2)$ , see also [5], [4] for the case of Bessel potential spaces. We define

$$\Phi(t) := e^t - \sum_{k=0}^{\lceil p-2 \rceil} \frac{t^k}{k!}, \quad (6)$$

where  $\lceil p-2 \rceil$  is the smallest integer greater than, or equal to  $p-2$ .

**Theorem 1.2.** *Let  $s \in (0, 1)$  and  $p > 1$  be such that  $sp = 1$ . There exists  $\alpha_* = \alpha_*(s) > 0$  such that for all  $\alpha \in [0, \alpha_*)$  it holds*

$$\sup_{u \in W^{s,p}(\mathbb{R}), \|u\|_{W^{s,p}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} \Phi(\alpha|u|^{\frac{1}{1-s}}) dx < \infty. \quad (7)$$

Moreover the supremum in (5) is infinite for any  $\alpha \in (\alpha^*, +\infty)$ , where  $\alpha^*$  is as in Theorem 1.1

As we shall see, Theorem 1.1 and 1.2 are sharp in the sense of (2). Indeed one of the open questions in [10] was whether an inequality of the type

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{\tilde{W}_0^{s,p}(I)} \leq 1} \int_I f(|u|) e^{\alpha|u|^{\frac{1}{1-s}}} dx < +\infty,$$

where  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is such that  $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$  holds true for the same exponents of the standard Moser-Trudinger inequality (see [4],[5]). For  $n = 1$  we prove the following

**Theorem 1.3.** *Let  $I \subset \mathbb{R}$  be a bounded interval,  $s \in (0, 1)$  and  $p > 1$  such that  $sp = 1$ . We have*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{\tilde{W}_0^{s,p}(I)} \leq 1} \int_I f(|u|) e^{\alpha^*|u|^{\frac{1}{1-s}}} dx = \infty, \quad (8)$$

$$\sup_{u \in W^{s,p}(\mathbb{R}), \|u\|_{W^{s,p}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} f(|u|) \Phi(\alpha^*|u|^{\frac{1}{1-s}}) dx = \infty, \quad (9)$$

where  $f: [0, \infty) \rightarrow [0, \infty)$  is any Borel measurable function such that  $\lim_{t \rightarrow +\infty} f(t) = \infty$ .

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## 2 Proof of Theorem 1.1

We start this section proving the validity of the Moser-Trudinger inequality (5). The result for  $n \geq 2$  is proved in [10] and the proof in the one dimensional case, which we report here for the sake of completeness, follows by a mild adaptation of the techniques in [10].

Thanks to [11, Theorem 9.1], using Sobolev embeddings and Hölder's inequality we have that there exists a constant  $C > 0$  independent of  $u$  such that for any  $u \in \tilde{W}_0^{s,p}(I)$

$$\|u\|_{L^q(\mathbb{R})} \leq C[u]_{W^{s,p}(\mathbb{R})} q^{1-s} \quad (10)$$

for any  $q > 1$ . For  $[u]_{W^{s,p}(\mathbb{R})} \leq 1$  we write

$$\int_I e^{\alpha|u|^{\frac{1}{1-s}}} dx = \sum_{k=0}^{\infty} \int_I \frac{\alpha^k}{k!} |u|^{\frac{k}{1-s}} dx \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{C}{1-s} \alpha k \right)^k, \quad (11)$$

where in the last inequality we used (10). Thanks to Stirling's formula

$$k! = \sqrt{2\pi k} \left( \frac{k}{e} \right)^k \left( 1 + O\left(\frac{1}{k}\right) \right) \quad (12)$$

the series in (11) converges for small  $\alpha$  and we recover a bound (uniform w.r.t.  $u$ ) for

$$\int_I e^{\alpha|u|^{\frac{1}{1-s}}} dx,$$

yielding (5).

As a direct consequence of (5), using the density of  $C_c^\infty(I)$  in  $\tilde{W}_0^{s,p}(I)$ , we have the following corollary (see [10, Proposition 3.2]).

**Corollary 2.1.** *If  $u \in \tilde{W}_0^{s,p}(I)$ , for every  $\alpha > 0$  it holds*

$$\int_I e^{\alpha|u|^{\frac{1}{1-s}}} dx < \infty.$$

We now give a useful result on the Gagliardo seminorm of radially symmetric functions (see [10, Proposition 4.3]), which will turn out to be useful later on.

**Proposition 2.1.** *Let  $u \in W^{s,p}(\mathbb{R})$  be radially symmetric and let  $sp = 1$ . Then*

$$[u]_{W^{s,p}(\mathbb{R})}^p = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy = 4 \int_0^{+\infty} \int_0^{+\infty} |u(x) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dx dy \quad (13)$$

*Proof.* The proof will follow from a direct computation. We split

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy + \int_{-\infty}^0 \int_{-\infty}^0 \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy \\ &+ \int_0^{+\infty} \int_{-\infty}^0 \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy + \int_{-\infty}^0 \int_0^{+\infty} \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy. \end{aligned}$$

Using a straightforward change of variable and the symmetry of  $u$ , we obtain the claim.  $\square$

To give an upper bound for the optimal exponent  $\bar{\alpha}$  such that the supremum in (5) is finite for  $\alpha \in [0, \bar{\alpha})$ , we define the family of functions

$$u_\varepsilon(x) := \begin{cases} |\log \varepsilon|^{1-s} & \text{if } |x| \leq \varepsilon \\ \frac{|\log |x||}{|\log \varepsilon|^s} & \text{if } \varepsilon < |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases} \quad (14)$$

Notice that the restrictions of  $u_\varepsilon$  to  $I$  belong to  $\tilde{W}_0^{s,p}(I)$ .

**Proposition 2.2.** *Let  $sp = 1$  and  $(u_\varepsilon) \subset \tilde{W}_0^{s,p}(I)$  be the family of functions defined in (14). Then*

$$\lim_{\varepsilon \rightarrow 0} [u_\varepsilon]_{W^{s,p}(\mathbb{R})}^p = \gamma_s := 8 \Gamma(p+1) \sum_{k=0}^{\infty} \frac{1}{(1+2k)^p}. \quad (15)$$

*Proof.* We will follow the proof in [10]. Define

$$I(\varepsilon) := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x-y|^2} dx dy. \quad (16)$$

Using Proposition 2.1 and (14) we see that  $I(\varepsilon)$  can be decomposed as

$$I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon),$$

where

$$I_1(\varepsilon) = \frac{8}{|\log \varepsilon|} \int_{\varepsilon}^1 \int_0^{\varepsilon} |\log x - \log \varepsilon|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dx dy,$$

$$I_2(\varepsilon) = \frac{4}{|\log \varepsilon|} \int_{\varepsilon}^1 \int_{\varepsilon}^1 |\log x - \log y|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dx dy,$$

$$I_3(\varepsilon) = 8 |\log \varepsilon|^{p-1} \int_1^{+\infty} \int_0^{\varepsilon} \frac{x^2 + y^2}{(x^2 - y^2)^2} dx dy,$$

$$I_4(\varepsilon) = \frac{8}{|\log \varepsilon|} \int_{\varepsilon}^1 \int_1^{+\infty} |\log x|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dx dy.$$

With an integration by parts, it is easy to check that  $\lim_{\varepsilon \rightarrow 0} I_i(\varepsilon) = 0$  for  $i = 1, 3, 4$ . As for  $I_2(\varepsilon)$ , integrating by parts after a change of variables we have

$$\begin{aligned} I_2(\varepsilon) &= \frac{4}{|\log \varepsilon|} \left\{ \log y \left( \int_{\frac{\varepsilon}{y}}^{\frac{1}{y}} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx \right) \right\} \Big|_{y=\varepsilon}^{y=1} \\ &\quad + \frac{4}{|\log \varepsilon|} \int_{\varepsilon}^1 \frac{\log y}{y^2} |\log \frac{1}{y}|^p \frac{\frac{1}{y^2} + 1}{\left(\frac{1}{y^2} - 1\right)^2} dy \\ &\quad - \frac{4\varepsilon}{|\log \varepsilon|} \int_{\varepsilon}^1 \frac{\log y}{y^2} |\log \frac{\varepsilon}{y}|^p \frac{\left(\frac{\varepsilon}{y}\right)^2 + 1}{\left(\left(\frac{\varepsilon}{y}\right)^2 - 1\right)^2} dy. \end{aligned}$$

A direct computation for the first term gives

$$\begin{aligned} &\frac{4}{|\log \varepsilon|} \left\{ \log y \left( \int_{\frac{\varepsilon}{y}}^{\frac{1}{y}} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx \right) \right\} \Big|_{y=\varepsilon}^{y=1} \\ &= 4 \int_1^{\frac{1}{\varepsilon}} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx, \end{aligned}$$

which converges to

$$4 \int_1^{+\infty} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx,$$

as  $\varepsilon \rightarrow 0$ . Moreover, since

$$\int_0^1 \frac{\log y}{y^2} \left| \log \frac{1}{y} \right|^p \frac{\frac{1}{y^2} + 1}{\left( \frac{1}{y^2} - 1 \right)^2} dy < +\infty$$

the second term in the sum converges to 0 as  $\varepsilon \rightarrow 0$ .

After setting  $\frac{\varepsilon}{y} = x$ , for the last term in the sum we have

$$\begin{aligned} & - \frac{4\varepsilon}{|\log \varepsilon|} \int_\varepsilon^1 \frac{\log y}{y^2} \left| \log \frac{\varepsilon}{y} \right|^p \frac{\left( \frac{\varepsilon}{y} \right)^2 + 1}{\left( \left( \frac{\varepsilon}{y} \right)^2 - 1 \right)^2} dy \\ &= - \frac{4}{|\log \varepsilon|} \int_\varepsilon^1 \log \left( \frac{\varepsilon}{x} \right) |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx \\ &= 4 \int_\varepsilon^1 |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx - \frac{4}{|\log \varepsilon|} \int_\varepsilon^1 |\log x|^{p+1} \frac{x^2 + 1}{(x^2 - 1)^2} dx \end{aligned}$$

which converges to

$$4 \int_0^1 |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx = 4 \int_1^{+\infty} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx$$

as  $\varepsilon \rightarrow 0$ . Summing up, we have

$$\lim_{\varepsilon \rightarrow 0} [u_\varepsilon]_{W^{s,p}(\mathbb{R})}^p = \lim_{\varepsilon \rightarrow 0} I_2(\varepsilon) = 8 \int_1^{+\infty} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx. \quad (17)$$

Integrating by parts we obtain

$$\begin{aligned} & \int_1^{+\infty} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx = p \int_1^{+\infty} \frac{|\log x|^{p-1}}{x^2 - 1} dx \\ &= p \int_0^1 \frac{|\log t|^{p-1}}{1 - t^2} dt, \end{aligned}$$

where we set  $t = \frac{1}{x}$ . Recall now

$$\frac{1}{1 - x^2} = \sum_{k=0}^{\infty} x^{2k}, \quad \int_0^1 |\log x|^{p-1} x^{2k} dx = \frac{\Gamma(p)}{(1 + 2k)^p}, \quad (18)$$

where  $\Gamma(\cdot)$  is the Euler Gamma function. Thanks to (18) we write

$$\int_0^1 \frac{|\log t|^{p-1}}{1 - t^2} dt = \sum_{k=0}^{\infty} \int_0^1 |\log t|^{p-1} t^{2k} dt = \Gamma(p) \sum_{k=0}^{\infty} \frac{1}{(1 + 2k)^p}, \quad (19)$$

proving (15). □

The upper bound for the optimal exponent follows directly from Proposition 2.2.

**Proposition 2.3.** *Let  $sp = 1$ . There exists  $\alpha^* := \gamma_s^{\frac{s}{1-s}}$  such that*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \leq 1} \int_I e^{\alpha|u|^{\frac{1}{1-s}}} dx = +\infty \quad \text{for } \alpha \in (\alpha^*, +\infty).$$

*Proof.* Let  $u_\varepsilon$  be the family of functions in  $\tilde{W}_0^{s,p}(I)$  defined in (14). Thanks to Proposition 2.2 we have that  $[u_\varepsilon]_{W^{s,p}(\mathbb{R})} \rightarrow (\gamma_s)^{\frac{1}{p}}$  as  $\varepsilon \rightarrow 0$ . Fix  $\alpha > \gamma_s^{\frac{s}{1-s}}$ . For  $\varepsilon$  small enough, there exists  $\beta > 0$  such that  $\alpha[u_\varepsilon]^{-\frac{1}{1-s}} \geq \beta > 1$ . If we set  $v_\varepsilon := \frac{u_\varepsilon}{[u_\varepsilon]}$  we have

$$\int_I e^{\alpha|v_\varepsilon|^{\frac{1}{1-s}}} dx \geq \int_{-\varepsilon}^{\varepsilon} e^{\alpha|v_\varepsilon|^{\frac{1}{1-s}}} dx \geq \int_{-\varepsilon}^{\varepsilon} e^{-\beta \log \varepsilon} dx = 2\varepsilon^{1-\beta} \rightarrow +\infty$$

as  $\varepsilon \rightarrow 0$ , since  $\beta > 1$ . □

### 3 Proof of Theorem 1.2

We shall adapt a technique by Ruf [13] to our setting.

For a measurable function  $u$  we set  $|u|^* : \mathbb{R} \rightarrow \mathbb{R}_+$  to be its non-increasing symmetric rearrangement, whose definition we shall now recall. For a measurable set  $A \subset \mathbb{R}$ , we define

$$A^* = (-|A|/2, |A|/2).$$

The set  $A^*$  is symmetric (with respect to 0) and  $|A^*| = |A|$ . For a non-negative measurable function  $f$ , such that

$$|\{x \in \mathbb{R} : f(x) > t\}| < \infty \quad \text{for every } t > 0,$$

we define the symmetric non-increasing rearrangement of  $f$  by

$$f^*(x) = \int_0^\infty \chi_{\{y \in \mathbb{R} : f(y) > t\}^*}(x) dt.$$

Notice that  $f^*$  is even, i.e.  $f^*(x) = f^*(-x)$  and non-increasing (on  $[0, \infty)$ ).

We will state here the two properties that we shall use in the proof of Proposition 1.2. The following one is proven e.g. in [7, Section 3.3].

**Proposition 3.1.** *Given a measurable function  $F : \mathbb{R} \rightarrow \mathbb{R}$  and a non-negative non-decreasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , it holds*

$$\int_{\mathbb{R}} F(f) dx = \int_{\mathbb{R}} F(f^*) dx.$$

The following Pólya-Szegő type inequality can be found e.g. in [2, Theorem 9.2].

**Theorem 3.1.** *Let  $0 < s < 1$  and  $u \in W^{s,p}(\mathbb{R})$ . Then*

$$[|u|^*]_W^{s,p}(\mathbb{R}) \leq [u]_W^{s,p}(\mathbb{R}).$$

Now given  $u \in W^{s,p}(\mathbb{R})$ , from Proposition 3.1 we get

$$\int_{\mathbb{R}} \Phi(\alpha(|u|)^{\frac{1}{1-s}}) dx = \int_{\mathbb{R}} \Phi(\alpha(|u|^*)^{\frac{1}{1-s}}) dx, \quad \| |u|^* \|_{L^p} = \|u\|_{L^p},$$

and according to Theorem 3.1

$$\| |u|^* \|_{W^{s,p}(\mathbb{R})}^p = \| |u|^* \|_{L^p(\mathbb{R})}^p + [ |u|^* ]_{W^{s,p}(\mathbb{R})}^p \leq \|u\|_{L^p(\mathbb{R})}^p + [u]_{W^{s,p}(\mathbb{R})}^p = \|u\|_{W^{s,p}(\mathbb{R})}^p.$$

Therefore in the rest of the proof of (7) we may assume that  $u \in W^{s,p}(\mathbb{R})$  is even, non-increasing on  $[0, \infty)$ , and  $\|u\|_{W^{s,p}(\mathbb{R})} \leq 1$ . We will use a technique by Ruf [13] (see also [5]) and write

$$\begin{aligned} & \int_{\mathbb{R}} \Phi(\alpha(|u|)^{\frac{1}{1-s}}) dx \\ &= \int_{I^c} \Phi(\alpha(|u|)^{\frac{1}{1-s}}) dx + \int_I \Phi(\alpha(|u|)^{\frac{1}{1-s}}) dx \\ &=: (I) + (II), \end{aligned}$$

where  $I = (-r_0, r_0)$ , with  $r_0 > 0$  to be chosen. Notice that since  $u$  is even and non-increasing, for  $x \neq 0$  and  $p > 1$ , we have

$$|u(x)|^p \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} |u(y)|^p dy \leq \frac{\|u\|_{L^p}^p}{2|x|}. \quad (20)$$

We start by bounding  $(I)$ . We observe that for  $r_0 \gg 1$ , we have  $|u(x)| \leq 1$  on  $I^c$  and hence

$$|u|^{\frac{p[p-1]}{p-1}} \leq |u|^p \quad \text{on } I^c,$$

since  $\frac{p[p-1]}{p-1} \geq p$ . For  $k > p-1$  we bound

$$\int_{I^c} (|u|^p)^{\frac{k}{p-1}} dx \leq \int_{I^c} \left( \frac{\|u\|_{L^p}^p}{2|x|} \right)^{\frac{k}{p-1}} = \frac{\|u\|_{L^p}^{\frac{pk}{p-1}} r_0^{1-\frac{k}{p-1}} (p-1)}{2^{\frac{k}{p-1}} (k+1-p)}.$$

Hence

$$\begin{aligned} (I) &= \sum_{k=\lceil p-1 \rceil}^{\infty} \int_{I^c} \frac{\alpha^k}{k!} |u|^{\frac{kp}{p-1}} dx \\ &= \frac{\alpha^{\lceil p-1 \rceil}}{\lceil p-1 \rceil!} \int_{I^c} |u|^{\frac{p[p-1]}{p-1}} dx + \sum_{k=\lceil p \rceil}^{\infty} \int_{I^c} \alpha^k \frac{|u|^{\frac{kp}{p-1}}}{k!} dx \\ &\leq C(\alpha, p) \|u\|_{L^p}^p + r_0(p-1) \sum_{k=\lceil p \rceil}^{\infty} \frac{\alpha^k (\|u\|_{L^p}^p)^{\frac{k}{p-1}}}{k!(k+1-p)(2r_0)^{\frac{k}{p-1}}} \\ &\leq C(\alpha, p) \|u\|_{L^p}^p + C \sum_{k=\lceil p \rceil}^{\infty} \left( \frac{\alpha}{(2r_0)^{p-1}} \right)^k \frac{1}{k!(k+1-p)} \leq C. \end{aligned}$$



As for (II), define  $v \in \tilde{W}_0^{s,p}(I)$  as follows

$$v(x) = \begin{cases} u(x) - u(r_0) & |x| \leq r_0 \\ 0 & |x| > r_0. \end{cases}$$

Let  $x \in I$ . We compute using the monotonicity of  $u$

$$\int_0^\infty |v(x) - v(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dy \leq \int_0^\infty |u(x) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dy. \quad (21)$$

Let  $x \in I^c$ . We have

$$\begin{aligned} & \int_0^\infty |v(x) - v(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dy \\ &= \int_I |u(r_0) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dy \\ &\leq \int_I |u(x) - u(y)|^p \frac{x^2 + y^2}{(x^2 - y^2)^2} dy. \end{aligned} \quad (22)$$

Combining (21), (22) and integrating in  $x$ , we get

$$[v]^p \leq [u]^p. \quad (23)$$

Using the definition of  $v$  and the inequality  $(a + b)^\sigma \leq a^\sigma + \sigma 2^{\sigma-1} (a^{\sigma-1} b + b^\sigma)$  for  $a, b \geq 0$  and  $\sigma \geq 1$ , we have

$$\begin{aligned} u^{\frac{1}{1-s}} &\leq v^{\frac{1}{1-s}} + \frac{1}{1-s} 2^{\frac{s}{1-s}} (v^{\frac{s}{1-s}} u(r_0) + u(r_0)^{\frac{1}{1-s}}) \\ &\leq v^{\frac{1}{1-s}} \left( 1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} \|u\|_p^p \right) + 2^{\frac{s}{1-s}} + \frac{2^{\frac{s}{1-s}}}{1-s} r_0 \\ &= v^{\frac{1}{1-s}} \left( 1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} \|u\|_p^p \right) + C(r_0). \end{aligned} \quad (24)$$

This implies

$$\begin{aligned} u(x) &\leq v(x) \left( 1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} \|u\|_p^p \right)^{1-s} + C^{1-s}(r_0) \\ &:= w(x) + C^{1-s}(r_0). \end{aligned}$$

From (23) and the definition of  $w$ , we get

$$\begin{aligned} [w]^p &= [v]^p \left( 1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} \|u\|_p^p \right)^{\frac{1-s}{s}} \\ &\leq (1 - \|u\|_p^p) \left( 1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} \|u\|_p^p \right)^{\frac{1-s}{s}} \end{aligned} \quad (25)$$

Consider now the function  $f(t) = (1-t)(1+\tau t)^\sigma$ , where  $\tau := \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)}$  and  $\sigma = \frac{1-s}{s} > 0$ . We compute

$$f'(t) = (1+\tau t)^{\sigma-1} (\tau t(-\sigma-1) + \tau\sigma - 1) \quad (26)$$

which vanishes for  $t_1 = -\frac{1}{\tau} < 0$  and  $t_2 = \frac{\tau\sigma-1}{\tau(\sigma+1)}$ . We choose now  $r_0 > 2^{\frac{2s-1}{1-s}}$  so that  $t_2 < 0$ . This implies that  $f$  is decreasing in  $(0, 1)$  and since  $f(0) = 1$  we have that  $f(t) < 1$  for  $t \in (0, 1)$ , which implies

$$[w]^p \leq 1. \quad (27)$$

We can apply now Proposition 1.1 on the interval  $I = (-r_0, r_0)$  to get that there exists  $\alpha_* > 0$  such that

$$\int_I e^{\alpha_* w^{p'}} dx \leq C \quad (28)$$

and using (24) we get

$$\int_I e^{\alpha_* u^{\frac{1}{1-s}}} dx \leq C \int_I e^{\alpha_* w^{\frac{1}{1-s}}} dx \leq C, \quad (29)$$

concluding the proof of (7).

To prove the second part of the claim one can argue as in the previous section, using the sequence of functions  $u_\varepsilon$  defined in (14) and taking into account that now the norm we are working with is the full  $W^{s,p}$ -norm. Indeed we have

$$\|u_\varepsilon\|_{L^p}^p = \int_{\mathbb{R}} |u_\varepsilon|^p dx = \int_{|x| \leq \varepsilon} (|\log \varepsilon|^{p-sp}) dx + \int_{\varepsilon < |x| < 1} \frac{|\log x|}{|\log \varepsilon|^{sp}} dx = O(|\log \varepsilon|^{-1}). \quad (30)$$

Hence from (15), it follows that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}^p = \gamma_s. \quad (31)$$

Choose  $M > 0$  large enough so that

$$\Phi(t) \geq \frac{1}{2}e^t, \quad t \geq M.$$

Then one has

$$\begin{aligned} \int_{\mathbb{R}} \Phi \left( \gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}} \right) dx &\geq \int_{u_\varepsilon \geq M} \Phi \left( \gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}} \right) dx \\ &\geq \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} e^{\left( \gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}} \right)} dx. \end{aligned} \quad (32)$$

for  $\varepsilon$  small enough. Now, thanks to (31), one can argue as in the proof of Proposition 2.3 to conclude the proof of Theorem 1.2.

## 4 Proof of Theorem 1.3

We will start by proving (8) since the proof of (9) will follow adapting the reasoning of the previous section.

Let  $u_\varepsilon$  be as in (14). To prove (8) it is enough to show that there exists a constant  $\delta > 0$  such that

$$\int_{-\varepsilon}^{\varepsilon} e^{\alpha^* \left( \frac{u_\varepsilon}{[u_\varepsilon]} \right)^{\frac{1}{1-s}}} dx \geq \delta.$$

Indeed,  $u_\varepsilon \rightarrow +\infty$  uniformly for  $|x| < \varepsilon$  as  $\varepsilon \rightarrow 0$  and we have

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \leq 1} \int_I f(|u|) e^{\alpha^* \left( \frac{|u|}{[u]} \right)^{\frac{1}{1-s}}} dx \geq \inf_{|x| < \varepsilon} f(|u_\varepsilon|) \int_{-\varepsilon}^{\varepsilon} e^{\alpha^* \left( \frac{|u_\varepsilon|}{[u_\varepsilon]} \right)^{\frac{1}{1-s}}} dx.$$

From Proposition 2.2, it follows that

$$\lim_{\varepsilon \rightarrow 0} \frac{[u_\varepsilon]}{\gamma_s^s} = 1 \quad (33)$$

and in particular

$$\lim_{\varepsilon \rightarrow 0} [u_\varepsilon]^p = 8 \int_1^{+\infty} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx = \gamma_s.$$

We compute

$$\lim_{\varepsilon \rightarrow 0} \log \frac{1}{\varepsilon} ([u_\varepsilon]^p - \gamma_s) = 8 \lim_{\varepsilon \rightarrow 0} \log \frac{1}{\varepsilon} \int_{\frac{1}{\varepsilon}}^{+\infty} |\log x|^p \frac{x^2 + 1}{(x^2 - 1)^2} dx = 0. \quad (34)$$

Then we can write

$$\frac{[u_\varepsilon]^p}{\gamma_s} \leq 1 + (C \log \frac{1}{\varepsilon})^{-1} \quad (35)$$

and in particular, recalling

$$\lim_{t \rightarrow +\infty} \frac{t}{(1 + \frac{C}{t})^{\frac{1}{1-s}}} - t = -\frac{1}{1-s},$$

we have

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} e^{\gamma_s^{\frac{s}{1-s}} \left( \frac{|u_\varepsilon|}{[u_\varepsilon]} \right)^{\frac{1}{1-s}}} dx &= \int_{-\varepsilon}^{\varepsilon} e^{\left( \frac{\gamma_s^s}{[u_\varepsilon]} \right)^{\frac{1}{1-s}} |u_\varepsilon|^{\frac{1}{1-s}}} dx \\ &\geq \int_{-\varepsilon}^{\varepsilon} e^{\frac{\log \frac{1}{\varepsilon}}{(1+C(\log \frac{1}{\varepsilon})^{-1})^{\frac{1}{1-s}}}} dx \\ &= 2\varepsilon e^{\frac{\log \frac{1}{\varepsilon}}{(1+C(\log \frac{1}{\varepsilon})^{-1})^{\frac{1}{1-s}}}} \rightarrow e^{-\frac{1}{1-s}} \end{aligned} \quad (36)$$

as  $\varepsilon \rightarrow 0$ . Therefore

$$\int_I e^{\gamma_s^{\frac{s}{1-s}} \left( \frac{|u_\varepsilon|}{[u_\varepsilon]} \right)^{\frac{1}{1-s}}} dx \geq \delta \quad (37)$$

for some  $\delta > 0$ , proving (8). We shall now prove (9). From (30) and (34) it follows that

$$\frac{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}^p}{\gamma_s} \leq 1 + O(|\log \varepsilon|^{-1}). \quad (38)$$

Now using (32) and arguing as in (36) and (37), we conclude the proof.

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